

HIGGS BUNDLES AND REPRESENTATION SPACES ASSOCIATED TO MORPHISMS

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ABSTRACT. Let G be a connected reductive affine algebraic group defined over the complex numbers, and $K \subset G$ be a maximal compact subgroup. Let X, Y be irreducible smooth complex projective varieties and $f : X \rightarrow Y$ an algebraic morphism, such that $\pi_1(Y)$ is virtually nilpotent and the homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is surjective. Define

$$\mathcal{R}^f(\pi_1(X), G) = \{\rho \in \text{Hom}(\pi_1(X), G) \mid A \circ \rho \text{ factors through } f_*\},$$

$$\mathcal{R}^f(\pi_1(X), K) = \{\rho \in \text{Hom}(\pi_1(X), K) \mid A \circ \rho \text{ factors through } f_*\},$$

where $A : G \rightarrow \text{GL}(\text{Lie}(G))$ is the adjoint action. We prove that the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0), G)//G$ admits a deformation retraction to $\mathcal{R}^f(\pi_1(X, x_0), K)/K$. We also show that the space of conjugacy classes of n almost commuting elements in G admits a deformation retraction to the space of conjugacy classes of n almost commuting elements in K .

1. INTRODUCTION

Let G be a connected reductive affine algebraic group defined over the complex numbers. Consider an algebraic morphism

$$f : X \rightarrow Y$$

where X and Y are irreducible smooth complex projective varieties, and let

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

be the induced morphism of fundamental groups, where $x_0 \in X$ is a base point. In certain situations, the representations

$$\rho : \pi_1(X, x_0) \rightarrow G$$

that factor through f_* have special geometric properties. See [KP], where necessary and sufficient conditions for such a factorization are given in terms of the spectral curve of the G -Higgs bundle associated to ρ .

In this article, we are interested in the whole moduli space of representations that factor in a similar way, and in its topological properties. Under some assumptions on f and Y , we provide a natural deformation retraction between two such representation spaces, described as follows.

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The Lie algebra of G will be denoted by \mathfrak{g} . Let $A : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the homomorphism given by the adjoint action of G on \mathfrak{g} . Fix a maximal compact subgroup $K \subset G$ and define:

$$\mathcal{R}^f(\pi_1(X, x_0), G) = \{\rho \in \mathrm{Hom}(\pi_1(X, x_0), G) \mid A \circ \rho \text{ factors through } f_*\},$$

$$\mathcal{R}^f(\pi_1(X, x_0), K) = \{\rho \in \mathrm{Hom}(\pi_1(X, x_0), K) \mid A \circ \rho \text{ factors through } f_*\}.$$

We note that the group G (respectively, K) acts on $\mathcal{R}^f(\pi_1(X, x_0), G)$ (respectively, on $\mathcal{R}^f(\pi_1(X, x_0), K)$) via the conjugation action of G (respectively, K) on itself. The quotient $\mathcal{R}^f(\pi_1(X, x_0), K)/K$ is contained in the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0), G)//G$.

We prove the following in Theorem 2.6:

Suppose that the fundamental group of Y is virtually nilpotent, and the homomorphism f_ is surjective. Then $\mathcal{R}^f(\pi_1(X, x_0), G)//G$ admits a deformation retraction to the subset $\mathcal{R}^f(\pi_1(X, x_0), K)/K$.*

In Section 3, we consider spaces of almost commuting elements in K and in G . Define:

$$\mathrm{AC}^n(K) = \{(g_1, \dots, g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_K \quad \forall i, j\},$$

where Z_K denotes the center of K . The moduli space of conjugacy classes:

$$\mathrm{AC}^n(K) / K,$$

where K acts by simultaneous conjugation, was studied in [BFM], [KS], and plenty of information is known in the cases $n = 2$ and $n = 3$. For instance, the number of components of $\mathrm{AC}^3(K) / K$ has been related in [BFM] to the Chern–Simons invariants associated to flat connections on a 3-torus.

In a similar fashion, we define $\mathrm{AC}^n(G)//G$, the moduli space of conjugacy classes of n almost commuting elements in G . For example, if G has trivial center, then $\mathrm{AC}^{2n}(G)//G$ coincides with

$$\mathrm{Hom}(\pi_1(X, x_0), G)//G,$$

where X is an abelian variety of complex dimension n . In Proposition 3.1, we show that $\mathrm{AC}^n(G) / G$ admits a deformation retraction to $\mathrm{AC}^n(K) / K$, and that the same holds for $\mathrm{AC}^n(G)$ and $\mathrm{AC}^n(K)$, extending one of the main results in [FL] and [BF1].

2. REPRESENTATION SPACES ASSOCIATED TO A MORPHISM

Let X be an irreducible smooth complex projective variety. Fix a point $x_0 \in X$. Let

$$f : X \longrightarrow Y$$

be an algebraic morphism, where Y is also an irreducible smooth complex projective variety, such that:

- (1) the fundamental group $\pi_1(Y, f(x_0))$ is virtually nilpotent, and
- (2) the homomorphism of fundamental groups induced by f

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)) \tag{2.1}$$

is surjective.

Using the homomorphism f_* in (2.1), we will consider $\pi_1(Y, f(x_0))$ as a quotient of the group $\pi_1(X, x_0)$.

Let G be a connected reductive affine algebraic group defined over \mathbb{C} . The Lie algebra of G will be denoted by \mathfrak{g} . Let

$$A : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \quad (2.2)$$

be the homomorphism given by the adjoint action of G on \mathfrak{g} . The affine algebraic variety (not necessarily irreducible) of representations

$$\rho : \pi_1(X, x_0) \longrightarrow G$$

will be denoted by $\mathrm{Hom}(\pi_1(X, x_0), G)$.

Definition 2.1. Let $\rho \in \mathrm{Hom}(\pi_1(X, x_0), G)$. We say that $A \circ \rho$ factors through f_* in (2.1) (or that $A \circ \rho$ *factors geometrically* through $f : X \rightarrow Y$, see [KP]) if there exists a homomorphism $\rho' \in \mathrm{Hom}(\pi_1(Y, f(x_0)), \mathrm{GL}(\mathfrak{g}))$ such that

$$\rho' \circ f_* = A \circ \rho. \quad (2.3)$$

Remark 2.2. (1) Clearly, if ρ itself factorizes as $\rho = \tilde{\rho} \circ f_*$ for some $\tilde{\rho} \in \mathrm{Hom}(\pi_1(X, x_0), G)$, then $A \circ \rho$ factorizes through f_* as in the definition; the converse is not always true.

(2) It is clear that $A \circ \rho \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}(\mathfrak{g}))$ factors through f_* as in (2.3), if and only if $A \circ \rho$ is trivial on the kernel of f_* . Moreover, when $A \circ \rho$ factors through f_* , a homomorphism $\rho' \in \mathrm{Hom}(\pi_1(Y, f(x_0)), \mathrm{GL}(\mathfrak{g}))$ satisfying equation (2.3) is unique, because f_* is surjective.

In the framework of non-abelian Hodge theory, there is a correspondence between semistable G -Higgs bundles over X and representations in $\mathrm{Hom}(\pi_1(X, x_0), G)$, [Si], [BG]. Denote by (E_ρ, θ_ρ) the semistable G -Higgs bundle on X associated to ρ under this correspondence. We note that (E_ρ, θ_ρ) is semistable with respect to every polarization on X .

Lemma 2.3. *Let $\rho \in \mathrm{Hom}(\pi_1(X, x_0), G)$ be such that $A \circ \rho$ factors through f_* . Then, the above principal G -bundle E_ρ on X is semistable.*

Proof. Let

$$\mathrm{ad}(E_\rho) := E_\rho \times^A \mathfrak{g} \longrightarrow X$$

be the adjoint vector bundle of E_ρ . The Higgs field on $\mathrm{ad}(E_\rho)$ induced by θ_ρ will be denoted by $\mathrm{ad}(\theta_\rho)$.

Let $\rho' : \pi_1(Y, f(x_0)) \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the unique homomorphism satisfying equation (2.3); the uniqueness of ρ' is a consequence of the surjectivity of f_* as remarked above. Let (E', θ') be the semistable Higgs vector bundle on Y associated to this homomorphism ρ' . Since the fundamental group of Y is virtually nilpotent, we know that the vector bundle E' is semistable [BF2, Proposition 3.1]. Let $c_i(E')$, $i \geq 0$, be the sequence of Chern classes of the bundle E' . Then, $c_i(E') = 0$ for all $i > 0$ because the C^∞ complex vector bundle underlying E' admits a flat connection (it is isomorphic to the C^∞ complex vector bundle underlying the flat vector bundle associated to ρ'). Therefore, by [BB, p. 39, Theorem 5.1], the vector bundle E' admits a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = E'$$

of holomorphic subbundles such that each successive quotient V_i/V_{i-1} , $1 \leq i \leq \ell$, admits a flat unitary connection. Consider the pulled back filtration

$$0 = f^*V_0 \subset f^*V_1 \subset \cdots \subset f^*V_{\ell-1} \subset f^*V_\ell = f^*E'. \quad (2.4)$$

A flat unitary connection on V_i/V_{i-1} pulls back to a flat unitary connection on

$$f^*V_i/(f^*V_{i-1}) = f^*(V_i/V_{i-1}).$$

Since each successive quotient for the filtration of f^*E' in (2.4) admits a flat unitary connection, we conclude that the holomorphic vector bundle f^*E' is semistable.

From (2.3) it follows that

$$(\text{ad}(E_\rho), \text{ad}(\theta_\rho)) = (f^*E', f^*\theta'). \quad (2.5)$$

Since f^*E' is semistable, from (2.5) it follows that $\text{ad}(E_\rho)$ is semistable. This implies that the principal G -bundle E_ρ is semistable [AB, p. 214, Proposition 2.10]. \square

Lemma 2.3 has the following corollary:

Corollary 2.4. *For any Higgs field θ , the G -Higgs bundle (E_ρ, θ) is semistable.*

Let

$$\rho^\lambda : \pi_1(X, x_0) \longrightarrow G \quad (2.6)$$

be a homomorphism corresponding to the Higgs G -bundle $(E_\rho, \lambda \cdot \theta_\rho)$, which is semistable by Corollary 2.4. We note that although ρ^λ is not uniquely determined by $(E_\rho, \lambda \cdot \theta_\rho)$, the point in the quotient space

$$\text{Hom}(\pi_1(X, x_0), G)/G$$

given by ρ^λ does not depend on the choice of ρ^λ . In other words, any two different choices of ρ^λ differ by an inner automorphism of the group G .

Lemma 2.5. *For every $\lambda \in \mathbb{C}$, the homomorphism $A \circ \rho^\lambda$ factors through f_* , where ρ^λ is defined in (2.6).*

Proof. Let $(\text{ad}(E_\rho)^\lambda, \text{ad}(\theta_\rho)^\lambda)$ be the Higgs vector bundle associated to the homomorphism $A \circ \rho^\lambda$. We note that $(\text{ad}(E_\rho)^\lambda, \text{ad}(\theta_\rho)^\lambda)$ is isomorphic to $(f^*E', f^*(\lambda \cdot \theta'))$, because the Higgs bundle (E', θ') corresponds to ρ' , and (2.3) holds. We saw in the proof of Lemma 2.3 that E' is semistable with $c_i(E') = 0$ for all $i > 0$. Since $(\text{ad}(E_\rho)^\lambda, \text{ad}(\theta_\rho)^\lambda)$ is isomorphic to the pullback of a semistable Higgs vector bundle on Y such that all the Chern classes of positive degrees of the underlying vector bundle on Y vanish, it can be deduced that $A \circ \rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$. In fact, if

$$\delta : \pi_1(Y, f(x_0)) \longrightarrow \text{GL}(\mathfrak{g})$$

is a homomorphism corresponding to the Higgs vector bundle $(E', \lambda \cdot \theta')$, then

- the homomorphism $A \circ \rho^\lambda$ factors through the quotient $\pi_1(Y, f(x_0))$, and
- the homomorphism $\pi_1(Y, f(x_0)) \longrightarrow \text{GL}(\mathfrak{g})$ resulting from $A \circ \rho^\lambda$ differs from δ by an inner automorphism of $\text{GL}(\mathfrak{g})$.

This completes the proof. \square

Fix a maximal compact subgroup

$$K \subset G.$$

Define

$$\mathcal{R}^f(\pi_1(X, x_0), G) = \{\rho \in \text{Hom}(\pi_1(X, x_0), G) \mid A \circ \rho \text{ factors through } f_*\},$$

$$\mathcal{R}^f(\pi_1(X, x_0), K) = \{\rho \in \text{Hom}(\pi_1(X, x_0), K) \mid A \circ \rho \text{ factors through } f_*\}.$$

Since $\pi_1(X, x_0)$ is a finitely presented group, the affine algebraic structure of G produces an affine algebraic structure on $\mathcal{R}^f(\pi_1(X, x_0), G)$. The group G acts on $\mathcal{R}^f(\pi_1(X, x_0), G)$ via the conjugation action of G on itself. Let

$$\mathcal{R}^f(\pi_1(X, x_0), G) // G$$

be the corresponding geometric invariant theoretic quotient. We note that this geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0), G) // G$ is a complex affine algebraic variety. Let

$$\mathcal{R}^f(\pi_1(X, x_0), K) / K$$

be the quotient of $\mathcal{R}^f(\pi_1(X, x_0), K)$ for the adjoint action of K on itself.

The inclusion of K in G produces an inclusion of $\mathcal{R}^f(\pi_1(X, x_0), K)$ in $\mathcal{R}^f(\pi_1(X, x_0), G)$, which, in turn, gives an inclusion

$$\mathcal{R}^f(\pi_1(X, x_0), K) / K \hookrightarrow \mathcal{R}^f(\pi_1(X, x_0), G) // G. \quad (2.7)$$

Instead of working with the Zariski topology on $\mathcal{R}^f(\pi_1(X, x_0), G) // G$, we consider on it the Euclidean topology which is induced from an embedding of this space in a complex affine space. Indeed, such an embedding can always be obtained by considering a finite set of generators of the algebra of G -invariant regular functions on $\mathcal{R}^f(\pi_1(X, x_0), G)$. Moreover, this topology is independent of the choice of such embedding, and compatible with the inclusion (2.7).

Theorem 2.6. *The topological space $\mathcal{R}^f(\pi_1(X, x_0), G) // G$ admits a deformation retraction to the above subset $\mathcal{R}^f(\pi_1(X, x_0), K) / K$.*

Proof. Two elements of $\text{Hom}(\pi_1(X, x_0), G)$ are called equivalent if they differ by an inner automorphism of G . Points of $\mathcal{R}^f(\pi_1(X, x_0), G) // G$ correspond to the equivalence classes of homomorphisms $\rho \in \text{Hom}(\pi_1(X, x_0), G)$ such that the action of $\pi_1(X, x_0)$ on \mathfrak{g} given by $A \circ \rho$ is completely reducible, meaning that \mathfrak{g} is a direct sum of irreducible $\pi_1(X, x_0)$ -modules. Let (E_ρ, θ_ρ) be the semistable G -Higgs bundle corresponding to the above homomorphism ρ , and let $(\text{ad}(E_\rho), \text{ad}(\theta_\rho))$ be the semistable adjoint Higgs vector bundle associated to (E_ρ, θ_ρ) . The above condition that the action of $\pi_1(X, x_0)$ on \mathfrak{g} given by $A \circ \rho$ is completely reducible is equivalent to the condition that the semistable Higgs vector bundle $(\text{ad}(E_\rho), \text{ad}(\theta_\rho))$ is polystable.

Let

$$\phi : (\mathcal{R}^f(\pi_1(X, x_0), G) // G) \times [0, 1] \longrightarrow \mathcal{R}^f(\pi_1(X, x_0), G) // G$$

be the map defined by $(\rho, \lambda) \longmapsto \rho^{1-\lambda}$ (defined in (2.6)), where $\rho \in \text{Hom}(\pi_1(X, x_0), G)$ satisfies the condition that the action of $\pi_1(X, x_0)$ on \mathfrak{g} given by $A \circ \rho$ is completely reducible. It is easy to see that ϕ is well-defined. We note that the point in the geometric invariant theoretic quotient $\mathcal{R}^f(\pi_1(X, x_0), G) // G$ given by ρ lies in the subset $\mathcal{R}^f(\pi_1(X, x_0), K) / K$ if and only if the Higgs field θ_ρ on the principal G -bundle E_ρ vanishes identically (as before, (E_ρ, θ_ρ) is the Higgs G -bundle corresponding to ρ).

The following are straightforward to check:

- $\phi(z, 0) = z$ for all $z \in \mathcal{R}^f(\pi_1(X, x_0), G) // G$,
- $\phi(z, 1) \in \mathcal{R}^f(\pi_1(X, x_0), K) / K$ for all $z \in \mathcal{R}^f(\pi_1(X, x_0), G) // G$, and
- $\phi(z, \lambda) = z$ for all $z \in \mathcal{R}^f(\pi_1(X, x_0), K) / K$ and $\lambda \in [0, 1]$.

Therefore, the above map ϕ produces a deformation retraction of $\mathcal{R}^f(\pi_1(X, x_0), G) // G$ to $\mathcal{R}^f(\pi_1(X, x_0), K) / K$. \square

Remark 2.7. Lemma 2.3 and Theorem 2.6 are also valid for morphisms $f : X \rightarrow Y$ in the category of compact Kähler manifolds, under the same assumptions on Y and f_* . The proofs of these results are analogous, by replacing semistability with the notion of *pseudostability* (see [BG], [BF2]).

3. DEFORMATION RETRACTION OF THE SPACE OF ALMOST COMMUTING ELEMENTS

Again, let G be a connected complex reductive group, and K be a maximal compact subgroup. Let

$$Z_G \subset G$$

be the center of G and let

$$PG := G / Z_G$$

be the quotient group. We note that the center of PG is trivial. Let

$$q : G \longrightarrow PG \tag{3.1}$$

be the quotient map. The image

$$PK := q(K) \subset PG$$

is a maximal compact subgroup of PG . We have $q^{-1}(PK) = K$.

Fix a positive integer n . Define

$$AC^n(G) = \{(g_1, \dots, g_n) \in G^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \ \forall i, j\}.$$

It is a subscheme of the affine variety G^n . The group G acts on $AC^n(G)$ as simultaneous conjugation of the n factors. Let

$$ACE^n(G) := AC^n(G) // G$$

be the geometric invariant theoretic quotient. Also, define

$$AC^n(K) = \{(g_1, \dots, g_n) \in K^n \mid g_i g_j g_i^{-1} g_j^{-1} \in Z_G \ \forall i, j\}.$$

So $AC^n(K) = AC^n(G) \cap K^n$. Let

$$ACE^n(K) := AC^n(K) / K$$

be the quotient for the simultaneous conjugation action of K on the n factors. Note that the inclusion of K in G produces an inclusion

$$ACE^n(K) \hookrightarrow ACE^n(G).$$

Proposition 3.1. *Let G be semisimple. Then, the topological space $ACE^n(G)$ admits a deformation retraction to the above subset $ACE^n(K)$.*

Proof. When G is semisimple, Z_G is a finite subgroup of G , so that the map (3.1) is a Galois covering. Also, $Z_G \subset K$. Define $\text{AC}^n(PG)$ and $\text{ACE}^n(PG)$ by substituting PG in place of G in the above constructions. Note that $\text{AC}^n(PG)$ parametrizes commuting n elements of PG because the center of PG is trivial. Similarly, define $\text{AC}^n(PK)$ and $\text{ACE}^n(PK)$ by substituting PK in place of K . So $\text{AC}^n(PK)$ parametrizes commuting n elements of PK . The projection

$$\beta : \text{ACE}^n(G) \longrightarrow \text{ACE}^n(PG) \quad (3.2)$$

constructed using the the projection q in (3.1) is a Galois covering with Galois group Z_G^n . However it should be mentioned that $\text{ACE}^n(G)$ need not be connected. Let

$$\gamma : \text{ACE}^n(K) \longrightarrow \text{ACE}^n(PK)$$

be the projection constructed similarly using q . Clearly, γ coincides with the restriction of β to $\text{ACE}^n(K) \subset \text{ACE}^n(G)$.

There is a deformation retraction of $\text{ACE}^n(PG)$ to $\text{ACE}^n(PK)$

$$\varphi : \text{ACE}^n(PG) \times [0, 1] \longrightarrow \text{ACE}^n(PG)$$

[FL, Theorem 1.1] (see also [BF1]). In particular, $\varphi|_{\text{ACE}^n(PG) \times \{0\}}$ is the identity map of $\text{ACE}^n(PG)$.

Applying the homotopy lifting property to the covering β in (3.2), there is a unique map

$$\tilde{\varphi} : \text{ACE}^n(G) \times [0, 1] \longrightarrow \text{ACE}^n(G)$$

such that

- (1) $\beta \circ \tilde{\varphi} = \varphi \circ (\beta \times \text{Id}_{[0,1]})$, and
- (2) $\tilde{\varphi}|_{\text{ACE}^n(G) \times \{0\}}$ is the identity map of $\text{ACE}^n(G)$.

This map $\tilde{\varphi}$ is a deformation retraction of $\text{ACE}^n(G)$ to $\text{ACE}^n(K)$, because φ is a deformation retraction. \square

Proposition 3.1 remains valid in the more general situation when G is reductive.

Theorem 3.2. *Let G be a connected reductive affine algebraic group over \mathbb{C} . Then, $\text{ACE}^n(G)$ admits a deformation retraction to the subset $\text{ACE}^n(K)$.*

Proof. First, note that Proposition 3.1 is clearly valid if G is a product of copies of the multiplicative group \mathbb{C}^* . Hence it remains valid for any G which is a product of a semisimple group and copies of \mathbb{C}^* . For a general connected reductive group G , consider the natural homomorphism

$$\eta : G \longrightarrow PG \times (G/[G, G]).$$

It is a surjective Galois covering map, the quotient $PG := G/Z_G$ is semisimple, while the quotient $G/[G, G]$ is a product of copies of \mathbb{C}^* . As mentioned above Proposition 3.1 is valid for $PG \times (G/[G, G])$. Using this and the above homomorphism η it follows that Proposition 3.1 is valid for G . \square

3.1. Deformation retraction of the space of n commuting elements. Finally, we note that the analogous result is also verified for the space of n commuting elements, $AC^n(G)$.

Theorem 3.3. *Let G be a connected reductive affine algebraic group over \mathbb{C} . Then, the space $AC^n(G)$ admits a deformation retraction to the subset $AC^n(K)$.*

Proof. Since PG and PK have trivial center, the spaces $AC^n(PG)$ and $AC^n(PK)$ consist of n commuting elements: If $(g_1, \dots, g_n) \in AC^n(PG)$, then

$$g_i g_j = g_j g_i, \quad \text{for all } i, j \in \{1, \dots, n\}.$$

Therefore, it is known that $AC^n(PG)$ admits a deformation retraction to $AC^n(PK)$ [PS, p. 2514, Theorem 1.1]. In view of this, imitating the proof of Proposition 3.1 it follows that $AC^n(G)$ admits a deformation retraction to $AC^n(K)$. \square

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